

MOEBIUS-WALSH CORRELATION BOUNDS AND AN ESTIMATE OF MAUDUIT AND RIVAT

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ABSTRACT. We establish small correlation bounds for the Moebius function and the Walsh system, answering affirmatively a question posed by G. Kalai [Ka]. The argument is based on generalizing the approach of Mauduit and Rivat [M-R] in order to treat Walsh functions of ‘large weight’, while the ‘small weight’ case follows from recent work due to B. Green [Gr]. The conclusion is an estimate uniform over the full Walsh system. A similar result also holds for the Liouville function.

§0. Introduction

Fix a large integer λ and restrict the Moebius function μ to the interval $[1, 2^\lambda] \cap \mathbb{Z} = \Omega$. Identifying Ω with the Boolean cube $\{0, 1\}^\lambda$ by binary expansion $x = \sum_{0 \leq j < \lambda} x_j 2^j$, the Walsh system $\{w_A; A \subset \{0, \dots, \lambda - 1\}\}$ is defined by $w_\emptyset = 1$ and

$$w_A(x) = \prod_{j \in A} (1 - 2x_j) = e^{i\pi \sum_{j \in A} x_j}. \quad (0.1)$$

The Walsh functions on Ω form an orthonormal basis (the character group of $(\mathbb{Z}/2\mathbb{Z})^\lambda$) and given a function f on Ω , we write

$$f = \sum_{A \subset \{0, \dots, \lambda - 1\}} \hat{f}(A) w_A \quad (0.2)$$

where $\hat{f}(A) = 2^{-\lambda} \sum_{n \in \Omega} f(n) w_A(n)$ are the Fourier-Walsh coefficients of f . Understanding the size and distribution of those coefficients is well-known to be important to various issues, in particular in complexity theory and computer science. Roughly speaking, a $F - W$ spectrum which is ‘spread out’ indicates a high level of complexity for the function f . We do not elaborate on this theory here and refer the reader to

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the extensive literature on the subject; see also the preprint of B. Green [Gr], which motivated this Note.

Returning to the Moebius function and the so-called ‘Moebius randomness law’ it seems therefore reasonable to expect that $\mu|_\Omega$ will have a $F - W$ spectrum that is not localized. More precisely, we establish the following uniform bound on its $F - W$ coefficients, answering affirmatively a question posed by G. Kalai.

Theorem 1. *For λ large enough,*

$$\max_{A \subset \{0, \dots, \lambda-1\}} \left| \sum_{n < 2^\lambda} \mu(n) w_A(n) \right| < 2^{\lambda - \lambda^{1/10}} \quad (0.3)$$

(a similar estimate is also valid for the Liouville function).

The proof of (0.3) involves different arguments, depending on the size $|A|$. Roughly speaking, one distinguishes between the case $|A| = o(\sqrt{\lambda})$ and $|A| \gtrsim \sqrt{\lambda}$. In the first case, B. Green already obtained an estimate of the type (0.3), see [Gr]. Part of the technique used in [Gr] is borrowed from Harman and Katai’s work [H-K] on prescribing binary digits of the primes. Let us point out that in this range the problem of estimating the correlation of μ with a Walsh function is reduced to estimates on the usual Fourier spectrum of μ (by an expansion of w_A in the trigonometric system). The latter is then achieved either by means of Dirichlet L -function theory (when the argument α is close to a rational $\frac{a}{q}$ with sufficiently small denominator q) or by Vinogradov’s estimate when q is large. At the other end of the spectrum, when $A = \{0, \dots, \lambda\}$, Mauduit and Rivat proved that

$$\left| \sum_{n < 2^\lambda} \Lambda(n) \widehat{w}_A(n) \right| < 2^{(1-\varepsilon)\lambda} \quad (0.4)$$

for some $\varepsilon > 0$.

Here $\Lambda(n)$ stands for the Van Mangold function ([M-R]). Their motivation was the solution to a problem of Gelfond on the uniform distribution of the sum of the binary digits of the primes. Of course, their argument gives a similar bound for the Moebius

function as well. Thus

$$\left| \sum_{n < 2^\lambda} \mu(n) \widehat{w}_{\{0, \dots, \lambda-1\}}(n) \right| < 2^{(1-\varepsilon)\lambda}. \quad (0.5)$$

A remarkable feature of the [M-R] method is that the usual type-I, type-II sum approach in the study of sums

$$\sum_{n < X} \Lambda(n) f(n) \text{ or } \sum_{n < X} \mu(n) f(n)$$

is applied directly to $f = w_{\{1, \dots, \lambda\}}$ without an initial conversion to additive characters (as done in [H-K] and [Gr]). The main idea in what follows is to generalize the Mauduit-Rivat argument in order to treat all Walshes w_A provided A is not too small (the latter case being captured by [Gr]).

Needless to say, the $2^{-\lambda^{1/10}}$ -saving in (0.3) can surely be improved (this is an issue concerning the treatment of low-weight Walsh functions) and no effort has been made in this respect. We also observe that, assuming *GRH*, (0.3) may be improved to

Theorem 2. *Under GRH, assuming λ large, we have*

$$\max_{A \subset \{0, \dots, \lambda-1\}} \left| \sum_{n < 2^\lambda} \mu(n) w_A(n) \right| < 2^{\lambda \left(1 - \frac{c}{(\log \lambda)^2}\right)}. \quad (0.6)$$

We will assume the reader familiar with the basic technique, going back to Vinogradov, of type-I and type-II sums, to which sums $\sum_{n < X} \mu(n) f(n)$ may be reduced; see [I-K] or [M-R]. In fact, we will rely here on the same version as used in [M-R] (see [M-R], Lemma 1). Otherwise, besides referring to the work of B. Green for $|A|$ small, our presentation is basically selfcontained. In particular, all the required lemmas pertaining to bounds on Fourier coefficients of Walsh functions are proven (they include estimates similar to those needed in [M-R] and also some additional ones) and are presented in §1 of the paper.

1. Estimates on Fourier coefficients of Walsh functions

For $A \subset \{0, \dots, \lambda - 1\}$ and $x = \sum_j x_j 2^j \in [1, 2^\lambda] \cap \mathbb{Z}$

$$w_A(x) = \prod_{j \in A} (1 - 2x_j) = e^{i\pi \sum_{j \in A} x_j} = \prod_{j \in A} h\left(\frac{x}{2^{j+1}}\right) \quad (1.0)$$

where $h : \mathbb{R} \rightarrow \{1, -1\}$ is the 1-periodic function

$$\begin{cases} h = 1 & \text{if } 0 \leq x < \frac{1}{2} \\ h = -1 & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

For $x \in \mathbb{Z}$,

$$h\left(\frac{x}{2^{j+1}}\right) = \sum_{|r| < 2^{j+1}} a_{r,j} e\left(\frac{rx}{2^{j+1}}\right) \text{ with } \sum |a_r| \lesssim j.$$

It follows that

Lemma 1. $w_A(x) = \sum_{k < 2^\lambda} \widehat{w}_A(k) e\left(\frac{kx}{2^\lambda}\right)$ with

$$\sum |\widehat{w}_A(k)| < (C\lambda)^{|A|}. \quad (1.1)$$

From the second equality in (1.0), also

$$\widehat{w}_A(k) = 2^{-\lambda} \sum_{\{x_j\}} e^{i\pi \sum_{j \in A} x_j} e^{2\pi i \frac{k}{2^\lambda} \sum_{j \notin A} x_j 2^j} = \prod_{j \notin A} \left(\frac{1 + e(k2^{j-\lambda})}{2} \right) \prod_{j \in A} \left(\frac{1 - e(k2^{j-\lambda})}{2} \right)$$

and

$$|\widehat{w}_A(k)| = \prod_{j \notin A} |\cos \pi k 2^{j-\lambda}| \prod_{j \in A} |\sin \pi k 2^{j-\lambda}| \quad (1.2)$$

Lemma 2.

$$\|\widehat{w}_A\|_\infty \lesssim 2^{-c|A|} \text{ for some constant } c > 0. \quad (1.3)$$

Proof. Use (1.2).

Taking some $i_0 \in A$ and assuming

$$\left| \sin \pi \frac{k}{2^{\lambda-i_0}} \right| \approx 1, \text{ hence } \left\| \frac{k}{2^{\lambda-i_0}} - \frac{1}{2} \right\| \approx 0$$

it follows that either

$$\left\| \frac{k}{2^{\lambda-i_0-1}} - \frac{1}{4} \right\| \approx 0$$

or

$$\left\| \frac{k}{2^{\lambda-i_0-1}} - \frac{3}{4} \right\| \approx 0$$

and in either case

$$\left| \cos \pi \frac{k}{2^{\lambda-i_0-1}} \right|, \left| \sin \pi \frac{k}{2^{\lambda-i_0-1}} \right| \sim \frac{1}{\sqrt{2}}.$$

The conclusion follows from (1.2). \square

In addition to (1.1), we have the bound

Lemma 3.

$$\sum_{k < 2^\lambda} |\widehat{w}_A(k)| \lesssim 2^{(\frac{1}{2}-c)\lambda}. \quad (1.4)$$

for some constant $c > 0$.

Proof. We have to estimate

$$\sum_{k \in \mathbb{Z}/2^\lambda \mathbb{Z}} \prod_{i \leq \lambda} \left| \cos \pi \left(\frac{u_i}{2} + \frac{k}{2^{\lambda-i}} \right) \right| \quad (1.5)$$

where $u_i = 1$ if $i \in A$ and $u_i = 0$ if $i \notin A$.

Perform a shift $k \rightarrow k + c2^{\lambda-2} + d2^{\lambda-1}$ with $c, d = 0, 1$.

This gives

$$\sum_{k \in \mathbb{Z}/2^{\lambda-2} \mathbb{Z}} \prod_{2 \leq i \leq \lambda} \left| \cos \pi \left(\frac{u_i}{2} + \frac{k}{2^{\lambda-i}} \right) \right|. \quad (*)$$

with

$$\begin{aligned}
(*) &= \frac{1}{4} \sum_{c,d=0,1} \left| \cos \pi \left(\frac{u_0}{2} + \frac{k}{2^\lambda} + \frac{c}{4} + \frac{d}{2} \right) \right| \left| \cos \pi \left(\frac{u_1}{2} + \frac{k}{2^{\lambda-1}} + \frac{c}{2} \right) \right| \\
&= \frac{1}{4} \sum_{c=0,1} \left(\left| \cos \pi \left(\frac{u_0}{2} + \frac{k}{2^\lambda} + \frac{c}{4} \right) \right| + \left| \sin \pi \left(\frac{u_0}{2} + \frac{k}{2^\lambda} + \frac{c}{4} \right) \right| \right) \left| \cos \pi \left(\frac{u_1}{2} + \frac{k}{2^{\lambda-1}} + \frac{c}{2} \right) \right| \\
&= \frac{1}{4} \left\{ (|\cos \phi| + |\sin \phi|) \cdot \left| \cos \left(\frac{\pi u_1}{2} + 2\phi \right) \right| + \right. \\
&\quad \left. \frac{1}{\sqrt{2}} (|\cos \phi - \sin \phi| + |\sin \phi + \cos \phi|) \cdot \left| \sin \left(\frac{\pi u_1}{2} + 2\phi \right) \right| \right\}
\end{aligned} \tag{1.6}$$

where $\phi = \pi \left(\frac{u_0}{2} + \frac{k}{2^\lambda} \right)$. Clearly

$$\begin{aligned}
(1.6) &\leq \frac{1}{4} \left\{ (1 + |\sin 2\phi|)^{\frac{1}{2}} \left| \frac{\cos}{\sin}(2\phi) \right| + (1 + |\cos 2\phi|)^{\frac{1}{2}} \left| \frac{\sin}{\cos}(2\phi) \right| \right\} \\
&\leq \frac{1}{4} \sqrt{2 + \sqrt{2}}.
\end{aligned}$$

Iterating, we obtain the bound

$$\leq \left(\sqrt{2 + \sqrt{2}} \right)^{\lambda/2}$$

and hence (1.4). \square

Lemma 4. *Let $r < \lambda$, $a = 0, 1, \dots, 2^r - 1$. Then*

$$\sum_{k \equiv a \pmod{2^r}} |\widehat{w}_A(k)| \lesssim 2^{(\frac{1}{2}-c)(\lambda-r)}. \tag{1.7}$$

Proof. Writing $k = a + 2^r k_1$ with $k_1 < 2^{\lambda-r}$,

$$\begin{aligned}
|\widehat{w}_A(k)| &= \prod_{i < \lambda-r} \left| \cos \pi \left(\frac{u_i}{2} + \frac{a}{2^{\lambda-i}} + \frac{k_1}{2^{\lambda-i-r}} \right) \right| \prod_{i \geq \lambda-r} \left| \cos \pi \left(\frac{u_i}{2} + \frac{a}{2^{\lambda-i}} \right) \right| \\
&\leq \prod_{i < \lambda-r} \left(\left| \cos \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right| + 2^{-\lambda+i+r} \right).
\end{aligned} \tag{1.8}$$

For fixed k_1 , denote

$$B(k_1) = \left\{ i < \lambda - r; \left| \cos \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right| < \left(\frac{1}{\sqrt{2}} \right)^{\lambda-r-i} \right\}$$

Hence, if $i \notin B_{k_1}$

$$\left| \cos \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right| + 2^{-\lambda+r+i} < \left(1 + \left(\frac{1}{\sqrt{2}} \right)^{\lambda-r-i} \right) \left| \cos \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right|$$

and if $i \in B_{k_1}$

$$\left| \cos \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right| + 2^{-\lambda+r+i} < \left(\frac{1}{\sqrt{2}} \right)^{\lambda-r-i} \left(1 + 2 \left(\frac{1}{\sqrt{2}} \right)^{\lambda-r-i} \right) \left| \sin \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right|.$$

Thus certainly

$$|\widehat{w}_A(k)| \lesssim \sum_{B \subset \{0,1,\dots,\lambda-r-1\}} \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda-r-i)} \prod_{\substack{i \notin B \\ i < \lambda-r}} \left| \cos \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right| \prod_{i \in B} \left| \sin \pi \left(\frac{u_i}{2} + \frac{k_1}{2^{\lambda-r-i}} \right) \right|. \quad (1.9)$$

Given $B \subset [0, \lambda - r - 1[$, define $B_1 \subset [0, \lambda - r - 1[$ as

$$B_1 = (B \cap [u_i = 0]) \cup (B^c \cap [u_i = 1]).$$

Hence

$$(1.9) = \sum_B \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda-r-i)} |\widehat{w}_{B_1}(k_1)|. \quad (1.10)$$

Summation of (1.10) over $k_1 < 2^{\lambda-r}$ and using the bound (1.4) with λ replaced by $\lambda - r$ clearly gives (1.7) \square

Next, we also need the following ‘approximation property’ for shifts

Lemma 5. *Let $A \subset [\lambda - \sigma, \lambda] \cap \mathbb{Z}$.*

Then

$$\sum_{k < 2^\lambda} |\widehat{w}_A(k)| < C^{(\log \lambda)^2} (2^\sigma)^{\frac{1}{2}-c}. \quad (1.11)$$

Moreover, there is a bounded function W_A on $[0, \lambda] \cap \mathbb{Z}$ satisfying $|\widehat{W}_A| \leq |\widehat{w}_A|$ and

$$(1.12) \quad \left(2^{-\lambda} \sum_{x < 2^\lambda} |W_A(x) - w_A(x)|^2 \right)^{1/2} < 2^{-ct}$$

$$(1.13) \quad \widehat{W}_A(k) = 0 \text{ if } |k| > 2^{\sigma+t}$$

Here $t \in \mathbb{Z}$ is a parameter satisfying $C(\log \lambda)^2 < t < \frac{1}{2}(\lambda - \sigma)$.

Proof. Writing $k = k_0 + 2^\sigma k_1$ with $k_0 < 2^\sigma$, $|k_1| < 2^{\lambda-\sigma-1}$ and setting again $u_i = 1$ if $i \in A$, $u_i = 0$ if $i \notin A$, we obtain

$$|\widehat{w}_A(k)| = \prod_{i < \lambda - \sigma} \left| \cos \pi \left(\frac{k_0 + 2^\sigma k_1}{2^{\lambda-i}} \right) \right| \cdot \prod_{\lambda - \sigma \leq i < \lambda} \left| \cos \pi \left(\frac{u_i}{2} + \frac{k_0}{2^{\lambda-i}} \right) \right| \quad (1.14)$$

$$= (1.14) \cdot |\widehat{w}_{A-\lambda+\sigma}(k_0)|. \quad (1.15)$$

where

$$A - \lambda + \sigma \subset [0, \sigma] \cap \mathbb{Z}.$$

We treat (1.14) as in the proof of Lemma 4, obtaining a bound

$$|(1.14)| < \sum_{B \subset \{0, 1, \dots, \lambda - \sigma - 1\}} \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda - \sigma - i)} |\widehat{w}_B(k_1)|. \quad (1.16)$$

From (1.1), certainly

$$\sum_{k_1 < 2^{\lambda-\sigma}} |\widehat{w}_B(k_1)| < (C\lambda)^{|B|} \quad (1.17)$$

and substitution of (1.17) in (1.16) implies by (1.15)

$$\|\widehat{w}_A\|_1 \leq \|\widehat{w}_{A-\lambda+\sigma}\|_1 \cdot \sum_B \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda - \sigma - i)} (C\lambda)^{|B|}$$

$$\stackrel{\text{Lemma 3}}{<} (2^\sigma)^{\frac{1}{2}-c} C^{(\log \lambda)^2}$$

which is (1.11).

Next, let $C(\log \lambda)^2 < \rho < \frac{1}{2}(\lambda - \sigma)$ and estimate

$$\sum_{k_1} \sum_{\min B \leq \lambda - \sigma - \rho} \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda - \sigma - i)} |\widehat{w}_B(k_1)| \lesssim 2^{-\rho/4}. \quad (1.18)$$

If

$$B \subset [\lambda - \sigma - \rho, \lambda - \sigma] \quad (1.19)$$

we establish a bound on $\widehat{w}_B(k_1)$. Write

$$|\widehat{w}_B(k_1)| = \prod_{i < \lambda - \sigma - \rho} \left| \cos \pi \frac{k_1}{2^{\lambda - \sigma - i}} \right| \cdot \prod_{\lambda - \sigma - \rho \leq i < \lambda - \sigma} \left| \cos \pi \left(\frac{v_i}{2} + \frac{k_1}{2^{\lambda - \sigma - i}} \right) \right|$$

with $v_i = 0, 1$ if $i \notin B$, $i \in B$. Hence, for $4^\rho < k_1 < 2^{\lambda - \sigma - 1}$

$$|\widehat{w}_B(k_1)| \leq \prod_{\rho < j \leq \lambda - \sigma} \left| \cos \pi \frac{k_1}{2^j} \right| < k_1^{-c} \quad (1.20)$$

for some $c > 0$, as we verify by dyadic expansion of k_1 .

It follows that for $4^\rho \leq K_1 < 2^{\lambda - \sigma}$

$$\begin{aligned} & \sum_{K_1 < |k_1| < 2^{\lambda - \sigma}} \left\{ \sum_{B(1.19)} \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda - \sigma - i)} |\widehat{w}_B(k_1)| \right\}^2 < \\ & < C \sum_{B(1.19)} \sum_{K_1 < |k_1| < 2^{\lambda - \sigma}} \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda - \sigma - i)} |\widehat{w}_B(k_1)|^2 \\ & \stackrel{(1.20)}{<} K_1^{-c} \sum_B \left(\frac{1}{\sqrt{2}} \right)^{\sum_{i \in B} (\lambda - \sigma - i)} \|\widehat{w}_B\|_1 \\ & \stackrel{(1.17)}{<} K_1^{-c} C(\log \lambda)^2. \end{aligned} \quad (1.21)$$

Define W_A as Fourier restriction of w_A . More specifically, let

$$W_A(x) = \sum \eta(k) \widehat{w}_A(k) e \left(\frac{kx}{2^\lambda} \right) \quad (1.22)$$

where $\eta : \mathbb{R} \rightarrow [0, 1]$ is trapezoidal with $\eta(z) = 1$ for $|z| < K_1 2^\sigma$, $\eta(z) = 0$ for $|z| \geq 2K_1 2^\sigma$. Hence $\|W_A\|_\infty \leq 3$ and $\widehat{W}_A(k) = \widehat{w}_A(k)$ for $|k| \leq K_1 2^\sigma$, $\widehat{W}_A(k) = 0$ for $|k| \geq 2K_1 2^\sigma$.

From the preceding

$$\begin{aligned} \|\widehat{W}_A - \widehat{w}_A\|_2^2 &\leq \sum_{k_0 < 2^\sigma} |\widehat{w}_{A-\lambda-\sigma}(k_0)|^2 \sum_{K_1 \leq |k_1| < 2^{\lambda-\sigma}} (1.16)^2 \\ &\stackrel{(1.18), (1.21)}{<} 2^{-\rho/2} + K_1^{-c} C^{(\log \lambda)^2}. \end{aligned} \quad (1.23)$$

Taking $K_1 = 2^{t-1}$, $\rho = \frac{t-1}{2}$, Lemma 5 follows. \square

The role of W_A is to provide a substitute for w_A with localized Fourier transform.

Lemma 6. *If $J \subset [1, 2^\lambda[$ is an interval, there is a bound*

$$\sum_{k \in J} |\widehat{w}_A(k)| \lesssim |J|^{\frac{1}{2}-c}. \quad (1.24)$$

Proof. Write

$$|\widehat{w}_A(k)| = \prod_{i < \lambda} \left| \cos \pi \left(\frac{u_i}{2} + \frac{k}{2^{\lambda-i}} \right) \right|$$

with $u_i = 0$ ($u_i = 1$) if $i \notin A$ ($i \in A$).

Assume $2^m \sim |J| < 2^m$. Obviously

$$\begin{aligned} |\widehat{w}_A(k)| &\leq \prod_{\lambda-m \leq i < \lambda} \left| \cos \pi \left(\frac{u_i}{2} + \frac{k}{2^{\lambda-i}} \right) \right| = \prod_{0 \leq i_1 < m} \left| \cos \pi \left(\frac{u_{i_1+\lambda-m}}{2} + \frac{k}{2^{m-i_1}} \right) \right| \\ &= |\widehat{w}_{A_1}(k)| \end{aligned}$$

where

$$A_1 = \{0 \leq i_1 < m; i_1 \in A + m - \lambda\}.$$

Hence, since \widehat{w}_{A_1} is 2^m -periodic

$$\sum_{k \in J} |\widehat{w}_A(k)| \leq \sum_{k \in J} |\widehat{w}_{A_1}(k)| \leq \sum_{k < 2^m} |\widehat{w}_{A_1}(k)| \leq \|\widehat{w}_{A_1}\|_1 < 2^{m(\frac{1}{2}-c)}$$

by Lemma 3. \square

2. Type-II sums

Let $X = 2^\lambda$, $S \subset \{0, \dots, \lambda-1\}$, $w_S(x) = \prod_{i \in S} (1 - 2x_i)$ with $x = \sum x_i 2^i$.

Specify ranges $M \sim 2^\mu$, $N \sim 2^\nu$ such that $M \leq N$ and $M.N \sim X$.

Our goal is to bound bilinear sums of the form $\sum_{\substack{m \sim M \\ n \sim N}} \alpha_m \beta_n w_S(m.n)$, where $|\alpha_m|, |\beta_n| \leq 1$ are arbitrary coefficients.

We fix a relatively small dyadic integer $L = 2^\rho$ (to be specified). We assume $\rho < \frac{\mu}{100}$, noting that otherwise our final estimate (2.29) is trivial.

Following [M-R], we proceed with the initial reduction of the problem, crucial to our analysis.

Estimate

$$\left| \sum_{\substack{m \sim M \\ n \sim N}} \alpha_m \beta_n w_S(m.n) \right| \leq \sum_{m \sim M} \left| \sum_{n \sim N} \beta_n w(m.n) \right|. \quad (2.1)$$

Fix K , such that $L2^K < N$ and write using Cauchy's inequality

$$\begin{aligned} \left| \sum_{n \sim N} \beta_n w(m.n) \right| &\leq \frac{1}{L} \sum_{n \sim N} \left| \sum_{\ell=1}^L \beta_{n+\ell 2^K} w(m(n+\ell 2^K)) \right| \\ \left| \sum_{n \sim N} \beta_n w(m.n) \right|^2 &\lesssim \frac{N}{L} \left[\sum_{\substack{n \sim N \\ |\ell| < L}} \beta_n \cdot \bar{\beta}_{n+\ell 2^K} w(m.n) w(m(n+\ell 2^K)) \right]. \end{aligned}$$

Hence, by another application of Cauchy's inequality, we obtain

$$(2.1)^2 \lesssim \frac{M.N}{L} \sum_{\substack{n \sim N \\ |\ell| < L}} \left| \sum_{m \sim M} w_S(m.n) w_S(m(n+\ell 2^K)) \right|. \quad (2.2)$$

Comparing the binary expansions of mn and $mn + \ell m 2^K$, the K first digits remain and we can assume that also digits $j > K + \mu + \rho + \varepsilon \rho$ are unchanged provided in (2.2) we introduce an additional error term of the order $2^{-\varepsilon \rho} M^2 N^2$ (cf. Lemma 5 in [M-R]). Here $\varepsilon > 0$ remains to be specified and we assume $\varepsilon \rho \in \mathbb{Z}_+$.

Therefore we may write, up to above error

$$w_S(mn) w_S(m(n+\ell 2^K)) \text{ '}' w_{S'}(mn) w_{S'}(m(n+\ell 2^K))$$

with

$$S' = S \cap [K, K + \mu + \rho'] \text{ and } \rho' = (1 + \varepsilon) \rho$$

and in (2.2) we may replace $w = w_S$ by $w_{S'}$.

We will either choose $K = 0$ or $\mu - \rho \leq K < \lambda - \mu - \rho$. Hence, by varying K , the intervals $[K, K + \mu + \rho]$ will cover $[0, \lambda]$.

For $K \neq 0$, we approximate $w_{S'}$ by $W_{S'}$ given by Lemma 5, applied with λ replaced by $K + \mu + \rho'$ and σ by $\mu + \rho'$.

Take $t = \varepsilon \rho$ where ρ is certainly assumed to satisfy

$$\frac{\mu}{100} > \rho \gg (\log \lambda)^2.$$

Thus from (1.12)

$$\sum_{x < X} |w_{S'}(x) - W_{S'}(x)|^2 < 2^{-ct} X.$$

From the preceding (since $W_{S'}$ is bounded)

$$(2.2) \lesssim \frac{X}{L} \sum_{\substack{n \sim N \\ |\ell| < L}} \left| \sum_{m \sim M} W_{S'}(m.n) W_{S'}(m(n + \ell 2^K)) \right| \quad (2.3)$$

$$\begin{aligned} &+ X \sum_{\substack{m \sim M \\ n \sim N}} |w_{S'}(mn) - W_{S'}(m.n)| \\ &+ X^2 L^{-\varepsilon} \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} (2.4) &< X \left(\sum_{x < X} |w_{S'}(x) - W_{S'}(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{x \in X} d(x)^2 \right)^{\frac{1}{2}} \\ &< L^{-c\varepsilon} X^2 (\log X)^C < L^{-c\varepsilon} X^2. \end{aligned}$$

For $K = 0$,

$$w_{S'}(x) = \sum_{k < 2^{\mu+\rho'}} \widehat{w}_{S'}(k) e\left(\frac{kx}{2^{\mu+\rho'}}\right) \quad (2.5)$$

where, from Lemma 2 and Lemma 3 applied with λ replaced by $\mu + \rho'$

$$\|\widehat{w}_{S'}\|_{\infty} < 2^{-c|S'|} \quad (2.6)$$

and

$$\|\widehat{w}_{S'}\|_1 < 2^{(\frac{1}{2}-c)(\mu+\rho')} < 2^{(\frac{1}{2}-c)(\mu+\rho)} \quad (2.7)$$

for ε small enough.

For $K \neq 0$,

$$W_{S'}(x) = \sum_{|k| < 2^{\mu+\rho'+t}} \widehat{W}_{S'}(k) e\left(\frac{kx}{2^{\mu+\rho'+K}}\right) \quad (2.8)$$

where

$$\|\widehat{W}_{S'}\|_{\infty} \leq \|\widehat{w}_{S'}\|_{\infty} < 2^{-c|S'|} \quad (2.9)$$

and by (1.11) and our choice of ρ

$$\|\widehat{W}_{S'}\|_1 < 2^{(\frac{1}{2}-c)(\mu+\rho)}. \quad (2.10)$$

Denoting by w either $w_{S'}$ when $K = 0$ or $W_{S'}$ for $\mu + \rho \leq K < \lambda - \mu - \rho$, substitution of (2.5), (2.8) and applying a smoothened m -summation gives for (2.3), with $M_1 = M^{1-\varepsilon_1}$

$$\frac{M^2 N}{L} \sum_{\substack{|\ell| \lesssim L \\ n \sim N}} \sum_{k, k'} |\widehat{w}(k)| |\widehat{w}(k')| 1_{\left[\left\|\frac{kn}{2^{\mu+\rho'+K}} - \frac{k'(n+\ell 2^K)}{2^{\mu+\rho'+K}}\right\| < \frac{1}{M_1}\right]} \quad (2.11)$$

up to a negligible error term.

The condition

$$\left\| \frac{(k-k')n}{2^{\mu+\rho'+K}} - \frac{k'\ell}{2^{\mu+\rho'}} \right\| < \frac{1}{M_1} \quad (2.12)$$

has to be analyzed.

For $k = k'$ the contribution is

$$\frac{M^2 N^2}{L} \sum_{|\ell| \lesssim L} \sum_{|k| < 2^{\mu+\rho'+t}} |\widehat{w}(k)|^2 1_{\left[\left\|\frac{k\ell}{2^{\mu+\rho'}}\right\| < \frac{1}{M_1}\right]}. \quad (2.13)$$

The $\ell = 0$ contribution in (2.2) is at most $\frac{M^2 N^2}{L}$.

For $\ell \neq 0$, we get a bound

$$M^{2+\varepsilon_1} N^2 2^{\rho'+t} \|\widehat{w}\|_{\infty}^2 < M^2 N^2 L^2 \|\widehat{w}\|_{\infty}^2 < X^2 L^2 2^{-c|S'|} \quad (2.14)$$

from (2.6), (2.9) and choosing $\varepsilon_1 > 0$ small enough to ensure $\varepsilon_1 \lambda < \varepsilon \rho$.

In the sequel, we assume $k \neq k', \ell \neq 0$.

Also, if in (2.11) for given k, k', ℓ there are at most $O(1)$ values of n satisfying (2.12), the resulting contribution is at most

$$M^2 N \|\widehat{w}\|_1^2 \underset{\substack{(2.7) \\ (2.10)}}{<} M^2 N (ML)^{1-2c} < X^2 L N^{-c} \quad (2.15)$$

since $M \leq N$.

Returning to (2.11), consider first the case $K = 0$.

We estimate the contribution for

$$(k - k', 2^{\mu+\rho'}) = 2^r.$$

Thus $k - k' = k_1 2^r$, $(k_1, 2) = 1$ and (2.12) becomes

$$\left\| \frac{k_1 n}{2^{\mu+\rho'-r}} - \frac{k' \ell}{2^{\mu+\rho'}} \right\| < \frac{1}{M_1} \quad (2.16)$$

implying also

$$\left\| \frac{k' \ell}{2^r} \right\| < \frac{L^{1+2\varepsilon}}{2^r}. \quad (2.17)$$

It follows from (2.17) that there are at most $L^{1+2\varepsilon}$ possibilities for $k' \pmod{2^r}$ and hence for $(k, k') \pmod{2^r}$.

For fixed k, k', ℓ , (2.16) determines $n \pmod{2^{\mu+\rho'-r}}$ up to $1 + L^{1+2\varepsilon} 2^{-r}$ possibilities and hence n up to $\frac{N 2^r}{ML} (1 + L^{1+2\varepsilon} 2^{-r})$ possibilities.

Thus the corresponding contribution to (2.11) is at most

$$\begin{aligned} & \frac{M^2 N}{L} \sum_{|\ell| \lesssim L} L^{1+2\varepsilon} \frac{N 2^r}{ML} (1 + L^{1+2\varepsilon} 2^{-r}) \max_a \sum_{\substack{k \equiv a \pmod{2^r} \\ k' \equiv a \pmod{2^r}}} |\widehat{w}(k)| |\widehat{w}(k')| \\ & \lesssim M N^2 (L + 2^r) L^{2\varepsilon} \max_a \left[\sum_{\substack{k < 2^{\mu+\rho'} \\ k \equiv a \pmod{2^r}}} |\widehat{w}(k)| \right]^2. \end{aligned} \quad (2.18)$$

From Lemma 4 applied with λ replaced by $\mu + \rho'$

$$\begin{aligned} (2.18) &\lesssim MN^2(L+2^r)(2^{\mu+\rho'-r})^{1-c}L^{2\varepsilon} \\ &= M^2N^2(L^22^{-r}+L)(ML2^{-r})^{-c}L^{3\varepsilon}. \end{aligned} \quad (2.19)$$

Hence, assuming

$$ML2^{-r} > L^C \quad (2.20)$$

we obtain the bound

$$\frac{X^2}{L}.$$

Next, assume

$$ML2^{-r} < L^C. \quad (2.21)$$

From the preceding, there are at most $L^{1+4\varepsilon}(ML2^{-r})^2 < L^C$ possibilities for (k, k') .

This gives the contribution

$$M^2N^2L^C\|\widehat{w}\|_\infty^2 < L^CX^22^{-c|S'|}$$

and in conclusion ($K = 0$) the bound

$$X^2(L^{-1} + L^C2^{-c|S'|}). \quad (2.22)$$

Next, assume

$$K \geq \mu - \rho. \quad (2.23)$$

Return to (2.11). Fix ℓ, k, k' with $|k - k'| \sim \Delta k < ML^2$. Letting n range over an interval of size $\frac{ML2^K}{\Delta k}$, the number of possibilities for n in that interval is at most

$$1 + \frac{L^{1+2\varepsilon}2^K}{\Delta k}.$$

Assume

$$N \gtrsim \frac{ML2^K}{\Delta k}.$$

The number of n 's satisfying (2.12) is at most (since $L2^K \geq M > \frac{\Delta K}{L^2}$ by (2.23))

$$\frac{N\Delta k}{ML2^K} \left(1 + \frac{L^{1+2\varepsilon}2^K}{\Delta k} \right) < \frac{N}{M}L^2.$$

This gives the contribution in (2.11)

$$L^2 MN^2 \|\widehat{w}\|_1^2 \underset{(2.10)}{<} L^2 MN^2 (ML^2)^{1-c} < X^2 L^3 M^{-c}. \quad (2.24)$$

Next, assume

$$N \ll \frac{ML2^K}{\Delta k}.$$

From (2.12), for ℓ, k, k' given, there are at most

$$1 + \frac{2^K L^3}{\Delta k} \sim \frac{2^K L^3}{\Delta k}$$

values of n .

Also

$$\left\| \frac{k'\ell}{2^{\mu+\rho'}} \right\| < \frac{1}{M_1} + \frac{\Delta k.N}{M.2^{\rho'}.2^K}.$$

Since $|k'\ell| < 2^{\mu+\rho} L^2$, there is some integer $\ell_1, |\ell_1| < L^2$ s.t.

$$\left| \frac{k'\ell}{2^{\mu+\rho'}} - \ell_1 \right| < \frac{1}{M_1} + \frac{\Delta k.N}{M.2^{\rho'}.2^K}$$

hence

$$\left| k' - \ell_1 \frac{2^{\mu+\rho'}}{\ell} \right| < L^{1+2\varepsilon} + \frac{\Delta k.N}{2^K}.$$

This restricts k' to at most L^2 intervals of size $L^{1+2\varepsilon} + \frac{\Delta k.N}{2^K}$.

Using Lemma 6, we obtain the following bound for the contribution to (2.11)

$$\begin{aligned} & M^2 N.L^2 \left(L^{1+2\varepsilon} + \frac{\Delta k.N}{2^K} \right)^{1-c} \frac{2^K L^3}{\Delta k} \lesssim \\ & \frac{M^2 N L^7 2^K}{\Delta k} + M^2 N^2 L^5 \left(\frac{\Delta k.N}{2^K} \right)^{-c} < M^2 N^2 L^7 \left(\frac{2^K}{N.\Delta k} \right)^c. \end{aligned} \quad (2.25)$$

If we assume

$$\frac{N.\Delta k}{2^K} > L^C$$

(2.25) gives the bound

$$\frac{X^2}{L}. \quad (2.26)$$

Assume next

$$\frac{N \cdot \Delta k}{2^K} < L^C.$$

From the preceding, k' is restricted to L^C values and the corresponding contribution to (2.11) is bounded by

$$M^2 N^2 L^C \|\widehat{w}\|_\infty^2 < X^2 L^C 2^{-c|S'|}. \quad (2.27)$$

Collecting previous bounds gives

$$(2.11) < X^2 \left(\frac{1}{L} + L^3 M^{-c} + L^C 2^{-c|S'|} \right) \quad (2.28)$$

and recalling (2.3), (2.4)

$$(2.1) < X \left(L^{-c\varepsilon} + L^2 M^{-c} + L^C 2^{-c|S'|} \right). \quad (2.29)$$

In the estimate (2.29), S' depends on the choice of K .

Recall that either $K = 0$ or $\mu - \rho \leq K < \lambda - \mu - \rho$ and hence, varying K , the intervals $[K, K + \mu + \rho]$ will cover $[0, \lambda - 1]$. Thus we may choose K as to ensure that

$$|S'| \geq \max |S \cap J| \gtrsim \frac{\mu}{\lambda} |S| \quad (2.30)$$

with max taken over intervals $J \subset [0, \lambda - 1]$ of size μ , in particular (2.29) implies

$$(2.1) < X \left(L^{-c\varepsilon} + L^2 M^{-c} + L^C 2^{-c \frac{\mu}{\lambda} |S|} \right) \quad (2.31)$$

where L is a parameter.

For $|S| \leq \frac{\lambda^{1/2}}{H}$ with $H \gg 1$ a parameter, we apply B. Green's estimate (see [Gr])

$$\left| \sum_{x < 2^\lambda} w_S(x) \mu(x) \right| < \lambda e^{-cH}. \quad (2.32)$$

Thus we assume $|S| > \frac{\lambda^{1/2}}{H}$. Taking $L = 2^H$, it follows from (2.29), (2.31) that

$$(2.1) \lesssim X \cdot 2^{-c\varepsilon H} \quad (2.33)$$

assuming either that

$$M > 2^{CH^2 \lambda^{1/2}} \quad (2.34)$$

or

$$M > C^H \text{ and } |S'| > CH \text{ } (S' \text{ satisfying (2.30)}). \quad (2.35)$$

3. Type-I sums and conclusion

We use Lemma 1 from [M-R] but treat also some of the type-I sums as type-II sums. Indeed, according to (2.33), (2.34), only the range $M < C^{H^2\lambda^{1/2}}$ remains to be treated.

Thus we need to bound

$$\sum_{m \sim M} \left| \sum_{n \sim N} w_S(mn) \right| \quad (3.1)$$

where $M.N \sim X = 2^\lambda$, $M < C^{H^2\lambda^{1/2}}$. We assume $|S| > \frac{\lambda^{1/2}}{H}$.

Expanding in Fourier and using a suitable mollifier in the n -summation, we obtain

$$(3.1) \leq \sum_{m \sim M} \sum_{k < X} |\widehat{w}_S(k)| \left| \sum_{n \sim N} e\left(\frac{kmn}{2^\lambda}\right) \right|$$

$$< N \sum_{\substack{m \sim M \\ k < X}} |\widehat{w}_S(k)| 1_{\left[\left\|\frac{km}{2^\lambda}\right\| < \frac{\lambda^2}{N}\right]} + o(1) \quad (3.2)$$

$$< NM^2 \lambda^2 \|\widehat{w}_S\|_\infty \quad (3.2')$$

$$< XM 2^{-c\lambda^{1/2}H^{-1}} \lambda^2. \quad (3.3)$$

Taking $H < \lambda^{1/10}$, (3.3) is certainly conclusive if $M < C^H$. Hence recalling (2.35), we can assume that

$$\mu > H \text{ and } \max |S \cap J| < CH \quad (3.4)$$

for any interval $J \subset \{0, \dots, \lambda - 1\}$ of size μ , where $M \sim 2^\mu$.

Assumption (3.4) will provide further information on \widehat{w}_S that will be useful in exploiting (3.2).

Write

$$S = S_1 \cup S_2$$

where $S_1 = S \cap [0, \lambda - 2\mu]$ and $S_2 = S \cap [\lambda - 2\mu, \lambda]$. Hence by (3.4),

$$|S_2| < CH.$$

Thus

$$\begin{aligned} w_{S_2}(x) &\stackrel{(1.0)}{=} \prod_{j \in S_2} h\left(\frac{x}{2^{j+1}}\right) \\ &= \sum_{k_2 \in \mathcal{A}_2} \widehat{w}_{S_2}(k_2) e\left(\frac{k_2 x}{2^\lambda}\right) + O_{L^1}(2^{-H}) \end{aligned} \quad (3.5)$$

where the set \mathcal{A}_2 may be taken of size

$$|\mathcal{A}_2| < 2^{H|S_2|} < C^{H^2} \quad (3.6)$$

(obtained by truncation of the Fourier expansion of h).

On the other hand

$$w_{S_1}(x) = \sum_{k_1 < 2^{\lambda-2\mu}} \widehat{w}_{S_1}(k_1) e\left(\frac{k_1 x}{2^{\lambda-2\mu}}\right)$$

and hence

$$w_S(x) = \sum_{\substack{k_1 < 2^{\lambda-2\mu} \\ k_2 \in \mathcal{A}_2}} \widehat{w}_{S_1}(k_1) \widehat{w}_{S_2}(k_2) e\left(\frac{2^{2\mu} k_1 + k_2}{2^\lambda} x\right) + O_{L^1}(2^{-H}). \quad (3.7)$$

The bound (3.2) becomes now

$$\begin{aligned} & N \sum_{\substack{m \sim M \\ k_1 < 2^{\lambda-2\mu} \\ k_2 \in \mathcal{A}_2}} |\widehat{w}_{S_1}(k_1)| |\widehat{w}_{S_2}(k_2)| \mathbf{1}_{\left[\left\|\frac{2^{2\mu} k_1 + k_2}{2^\lambda} m\right\| < \frac{\lambda^2}{N}\right]} \\ & < N |\mathcal{A}_2| \|\widehat{w}_{S_1}\|_\infty \max_{k_2} \sum_{m \sim M} \left| \left\{ k_1 < 2^{\lambda-2\mu}; \left\|\frac{2^{2\mu} k_1 + k_2}{2^\lambda} m\right\| < \frac{\lambda^2}{N} \right\} \right|. \end{aligned} \quad (3.8)$$

Clearly

$$\begin{aligned} & \sum_{m \sim M} \left| \left\{ k_1 < 2^{\lambda-2\mu}; \left\|\frac{k_1 m}{2^{\lambda-2\mu}}\right\| < \frac{2\lambda^2}{N} \right\} \right| = \\ & \sum_{m \sim M} \left| \left\{ k_1 < 2^{\lambda-2\mu}; k_1 m \equiv 0 \pmod{2^{\lambda-2\mu}} \right\} \right| \lesssim \mu \cdot M \end{aligned}$$

and therefore, since $|S_1| \gtrsim \frac{\lambda^{1/2}}{H}$ and (3.6)

$$(3.8) < \mu C^{H^2} 2^{-c\lambda^{1/2}H^{-1}} NM$$

$$< 2^{-c\lambda^{1/2}H^{-1}}X. \quad (3.9)$$

From (2.33) and (3.9), we can claim a uniform bound

$$\left| \sum_{x < X} \mu(x) w_S(x) \right| \lesssim X \cdot 2^{-c\lambda^{1/10}} \quad (3.10)$$

hence obtaining Theorem 1.

Under GRH, (3.10) can be improved of course.

First, from a result due to Baker and Harman [B-H], there is a uniform bound

$$\left\| \sum_{n \in X} \mu(n) e(n\theta) \right\|_{\infty} \ll X^{\frac{3}{4} + \varepsilon}. \quad (3.11)$$

Hence

$$\left| \sum_{n < X} \mu(n) w_S(n) \right| < \|\hat{w}_S\|_1 X^{\frac{3}{4} + \varepsilon'} < (\log X)^{|S|} X^{\frac{3}{4} + \varepsilon'} \quad (3.12)$$

and we may assume

$$|S| > c \frac{\log X}{\log \log X}. \quad (3.13)$$

If (3.13), apply the type-I-II analysis above.

From (2.31), assuming

$$M \sim 2^\mu > X^{c_1 \frac{1}{\log \log X}} \quad (3.14)$$

and choosing L appropriately, we obtain

$$(2.1) < X \cdot 2^{-c \frac{\log X}{(\log \log X)^2}}. \quad (3.15)$$

If M fails (3.14) the type-I bound (3.2') gives

$$\begin{aligned} (3.1) &< X \cdot M \|\hat{w}_S\|_{\infty} \\ &\stackrel{(1.3)}{<} X \cdot X^{c_1 \frac{1}{\log \log X}} 2^{-c' \frac{\log X}{\log \log X}} \\ &< X^{1 - c_2 \frac{1}{\log \log X}} \end{aligned} \quad (3.16)$$

for appropriate choice of c_1 in (3.14).

In either case

$$\left| \sum_{n < X} \mu(n) w_S(n) \right| < X^{1 - \frac{\epsilon}{(\log \log X)^2}} \quad (3.17)$$

which is Theorem 2.

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